A REVIEW OF LITERATURE RELATING TO ORTHOGONAL ARRAYS

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ABSTRACT

Orthogonal arrays are statistical methods, the terminology of which comes from experimental design. Orthogonal arrays can be expanded to others experimental designs. Therefore, this paper presents a review of the literature available on orthogonal arrays.

Key Words : Orthogonal Array, Orthogonal Latin Squares, Hadamard Matrices, Projective Geometry, Compound Orthogonal Array.

ORTOGONAL DÜZENLER İLE İLGİLİ LİTERATÜR ARAŞTIRMASI

ÖZET

Ortogonal düzenler deney düzenlemelerinde kullanılan istatistiksel bir tekniktir ve ortogonal düzenler diğer deneme düzenlerine genişletilebilir. Bu makalede ortogonal düzenler için literatürde mevcut olan çalışmalar araştırıldı.

Anahtar Kelimeler : Ortogonal Düzen, Ortogonal Latin Kare, Hadamard Matrisleri, Projektif Geometri, Birleşik Ortogonal Düzen.

1. INTRODUCTION

An orthogonal array of size N, k constraints, s levels and strength t, denoted OA(N,k,s,t), is a $k \times N$ matrix X of s symbols such that all the ordered t-tuples of the symbols occur equally often as column vectors of any $t \times N$ submatrix of X. It is clear that N must be of the form λs^t, where λ is usually called the index of the orthogonal array. In applications to factorial designs, each row corresponds to a factor, the symbols are factor levels and each column represents a combination of the factor levels. Thus every OA(N,k,s,t) defines an N-run factorial design for k factors each having s levels (21). Where λ=1 and refer to such arrays as “orthogonal arrays of index unity” (15).

Sylvester (49) was considered orthogonal arrays with t=2, s=2 who gives an explicit construction for the case $N=2^m$ (47).

Rao (39, 40) first introduced the concept of orthogonal arrays. He discussed the use of these arrays as fractionally replicated plans for symmetrical factorial experiments which permit the estimation of main-effects and interactions up to order (t-2) when higher order interactions are negligible (1). He studied hypercubes of strength t, which are orthogonal arrays for which the index λ is a power of s. He has used them in connection with confounded factorial designs (11). Rao first introduced orthogonal arrays into statistics under the name of hypercubes (47).

Rao generalized the concepts further to obtain hypercubes of strength t that are equivalent to Latin and orthogonal Latin cubes and hypercubes of the first order when t=2. Furthermore, Rao considered constructions of hypercubes from PG(t,s). The present notion of orthogonal arrays is also due to Rao (40) and hypercubes of strength t form a subclass of orthogonal arrays (38).

Paley (35) first systematically studied the construction $H_{4\lambda}$ known in mathematical literature as Hadamard matrices (38). Paley was interested in orthogonal arrays with t=2, s=2 because of their applications to the theory of polytopes (47).

Butson (16, 17) considered generalized Hadamard matrices (48).

Shrikhande (48) showed that some connections between generalized Hadamard matrices and various tactical configurations such as group divisible and orthogonal arrays of strength two. Some constructions for these arrays are also indicated.

Plotkin (37) makes the very strong conjecture that every Hadamard matrix of order 8n can be obtained from specializing some orthogonal design of order 8n. He showed that the existence of a Hadamard matrix of order n implies the existence of three types of orthogonal designs (23).

Hedayat and Wallis (23) showed that there are many of results on constructions and applications of Hadamard matrices.

Bose and Kishen (8) first developed an established connection between orthogonal arrays and the theory of confounding in symmetrical factorial designs (based on the use of finite projective geometries), later amplified by Bose (11).

Bose (10) proved that there are a maximum number of factors, which can be accommodated in symmetrical factorial designs. Furthermore, Bose showed that certain statistical problems of design of experiments could be attacked fruitfully by interpreting the statistical terms involved in terms of finite geometries. Seiden (45) showed that on finite geometry explains shortly the PG(3,4). Bose has shown the maximum number of factors that it is possible to accommodate in a symmetrical factorial experiment (1).

Bush (15) proved and proceeded to generalize the notion of a set of orthogonal Latin squares and whose term this extension is an orthogonal array of index unity. Bush secure bounds for the number of mutually orthogonal Latin squares of side s are bounded above by s-1. Furthermore, Bush gives a method of constructing these arrays by considering a class of polynomials with coefficients in the finite Galois field GF(s), where s is a prime or power of a prime.

Bose and Nair (6) established a connection between orthogonal arrays and partially balanced designs (11).

Bose (7) used the method of differences has been elsewhere for constructing incomplete block designs. Later Bose and Bush (11) use it to construct orthogonal arrays of strength 2 (11).

Bose (9) gave a number of arrays of strength 2 for other values of s, and established a connection between orthogonal arrays and affine resolvable balanced incomplete block designs (11).

Kishen (29, 30) generalized the concept of Latin squares and orthogonal Latin squares to Latin cubes and hypercubes and orthogonal Latin cubes and hypercubes.

Kishen introduced the concept of Latin hypercubes and orthogonal Latin hypercubes (38).

Plackett and Burman (36) showed that methods are developed to avoid the complete factorial experiment in industrial experimentation when the number of factors are so large that the standard procedure is impracticable. By assuming a simplified linear hypothesis, the problems of determining main effects with maximum precision are reduced to a combinatorial one. Practically all-useful solutions for this have been found when each factor appears at two levels, but the solutions for more than two levels are fairly limited. The relationship of these solutions to some encountered in balanced incomplete blocks has been discussed. They obtained the designs for two levels and all possible $N \leq 100$ assemblies (except $N=92$).

Plackett and Burman studied orthogonal arrays in their terminology multifactorial designs, from the point of view of an application in physical and industrial research (46).

Plackett and Burman considered that main-effect plans fall within the class of optimum multifactorial designs (1). Plackett and Burman applied Paley's work to their research in physics and industry (47).

Rao (40) discussed the use of orthogonal arrays together with some methods of constructing them and gives the generalization of the inequality of Plackett and Burman (11).

Kempthorne (28) analyzed Plackett and Burman designs (46).

Hotelling (27) considered orthogonal arrays of strength two and two levels from the point of application of factorial designs to chemistry (46). Hotelling suggested that the work of Paley solved the problem of weighing design (47). Mood (33) continued the work of Paley and Hotelling (47).

Fisher (22) developed the main-effect plans derivable from the system of confounding that can be represented by the orthogonal arrays (1).

Rao (40, 41) orthogonal arrays of strength 2 are examples of orthogonal main-effect plans with the added advantage that comparable main-effect contrasts for different factors are estimated with the same efficiency (25).

Bose (12) formulated for the purpose of constructing orthogonal arrays of strength 2. It is shown further that each of the multi-factorial designs of Plackett and Burman in which each factor takes two levels (46).

Bose and Bush (11) showed that orthogonal arrays can be regarded as natural generalizations of orthogonal Latin squares, and are useful in various problems of experimental design. In that paper the known upper bounds for the maximum possible number of constraints for arrays of strength 2 and 3 have been improved and certain methods for constructing these arrays have been given. They pointed out the optimum multi-factorial designs considered by Plackett and Burman is essentially orthogonal arrays of strength 2.

Kishen (30) showed that the maximal number of mutually orthogonal $s\times s\times s$ cubes of first order is s^2+s-2 and gave methods of constructing them when s is a prime or prime power. Kishen given the construction of such cubes when s is prime or a prime power (38).

Seiden (46) formulated a method of constructing orthogonal arrays of an arbitrary strength t. This method is a modification of the method based on differences. However, an orthogonal array can be constructed of arrays of strength 3 from arrays of strength 2. An orthogonal array OA(36,13,3,2) is constructed. The method used for its construction cannot lead to a number of constraints greater than 13.

Seiden has shown one can construct orthogonal arrays OA($\lambda 2^3$,k+1,2,3) from arrays OA($\lambda 2^2$,k,2,2) with the maximum number of constraints k+1 provided that k is the maximum number of constraints for the arrays of strength two (47).

Addelman and Kempthorne (1) presented a method of constructing main-effect plans for symmetrical factorial experiments. As main-effect plans are orthogonal arrays of strength two the method presented permits the construction of the orthogonal arrays. The plans for fractionally replicated symmetrical factorial experiments, which are developed in that paper, are orthogonal arrays of strength two. They named the plans main-effect plans because they permit orthogonal estimation of all the main-effects when the interactions are negligible.

Addelman and Kempthorne (2) conducted an extensive study of orthogonal main-effect plans. Their report contains various useful ideas for the construction of such plans, some of which are made more accessible and further extended in Addelman and Kempthorne (1) and Addelman (3, 4). The catalogue of orthogonal main-effect plans in Addelman and Kempthorne (2) contains also various asymmetrical orthogonal arrays (25).

Bose (13) applied orthogonal arrays to information theory and pointed out the analogy between the problems of design of experiments and information theory (47).

Chakravarti (18) generalized the concept of orthogonal arrays to partially balanced arrays (38).

Chakravarti (19) showed that construction of orthogonal arrays and determined relationship orthogonal arrays and partially balanced arrays. Chakravarti gave properties of the orthogonal Latin squares.

Seiden and Zemach (47) showed that the orthogonal arrays OA($\lambda 2^{t+1}$,k+1,2,t+1) could be constructed from the arrays OA($\lambda 2^t$,k,2,t) with the maximum number of constraints if k is the maximum number of constraints of strength t. The particular case of this result for t=2 was given by Seiden (46).

Seiden and Zemach generalized the construction of arrays OA($\lambda 2^{t+1}$,k+1,2,t+1) from arrays OA($\lambda 2^t$,k,2,t). The structure of arrays OA($\lambda 2^t$,t+1,2,t) is analyzed for λ=q2^n, q odd, a method of extending any array OA($\lambda 2^t$,t+1,2,t) to t+n+1 constraints is described. Orthogonal arrays OA($\lambda 2^4$,k,2,4) are discussed in detail for λ=1 through λ=5. The maximum value of k is established in each of these cases and arrays assuming these values are effectively constructed.

Blum, Schatz and Seiden (5) showed that the maximum possible number of constraints for a 2-level orthogonal array of odd index with strength t is t+1. They pointed out the terminology comes from experimental design.

Raghavarao (38) gave the definition of the orthogonal array, different construction methods and maximum the number of constraints. Further, Raghavarao showed that projective geometries relating to orthogonal arrays, product of orthogonal arrays and relationship between orthogonal arrays and partially balanced arrays.

Chakravarty and Dey (20) used some patchwork methods involving incidence matrices of balanced incomplete block designs to construct orthogonal arrays of strength 2 and 3 (24).

Hedayat and Stufken (24) results on the construction and maximum number of factors in orthogonal arrays are discussed, with emphasis on arrays in which the factors are at two levels.

Ray-Chadhuri and Singhi (42) showed that a signed orthogonal array SOA(N,k,s) exists for all (N,k,s,t),k≥t and an orthogonal array OA(N,k,s) exist for all OA(N,k,s,t) provided s is sufficiently large. A reciprocity relation for the number of distinct orthogonal array is derived.

Roberts (43) studied sets of mutually orthogonal Latin squares with “like sub-squares”. Roberts introduced the concepts of “transversal identifying Latin squares” and Latin squares with “like sub-squares”.

Hedayat, Pu and Stufken (25) presented general techniques for the construction of asymmetrical orthogonal arrays of strength 2. These are then applied to special cases to obtain new families of such arrays. Among these are saturated main-effect plans based on s^m runs with factors at s^{ν_i} levels, i=0,1,…,r, where $m \geq \nu_r$, $\nu_0=1$, ν_{i-1} divides ν_i, i=1,2,…,r, and s is a prime power.

The current emphases on quality control and product improvement have rejuvenated research in the area of factorial design. They presented and combined techniques for the construction of arguably the most appealing asymmetrical factorial designs, namely asymmetrical orthogonal arrays.

Mukerjee and Kageyama (34) an orthogonal array OA(N,k,s,t) attaining the Rao(40) (derivable from combinatorial arrangements of arrays) bound on parameters is said to be complete. They investigated complete orthogonal arrays with s=2 symbols. The cases t=6,7 are completely characterized. In addition, for 8≤t≤13, complete orthogonal arrays are almost entirely characterized.

Lin and Draper (31, 32) studied projections of Plackett-Burman designs. Their computer searches found all the projections of 12-, 16-, 20-, 24-, 28-, 32-, and 36- run Plackett-Burman designs onto three factors. Lin and Draper further considered projections of 12-, 16-, 20-, and 24- run Plackett-Burman designs onto four and five factors (21). Lin and Draper the projection properties of the 2_R^{q-p} fractional factorials are well known and have been used effectively in a number of published examples of experimental investigations. The Plackett and Burman designs also have interesting projective properties, knowledge of which allows the experimenter to follow up an initial Plackett and Burman design runs that increase the initial resolution for the factors that appear to matter and thus permit efficient separation of effects to interest. Projections of designs into 2-5 dimensions are discussed, and the 12-run case is given in detail. A numerical example illustrates the practical uses of these projections.

Cheng (21) the definition of an orthogonal array imposes an important geometric property: the projection of an $OA(\lambda 2^t, 2^k, t)$, a $\lambda 2^t$-run orthogonal array with k two level factors and strength t, onto any t factors consists of λ copies of the complete 2^t factorial. Cheng considered projections of an $OA(N, 2^k, t)$ onto t+1 and t+2 factors. An important projection property of an orthogonal array: when projected onto any t factors, it yields λ copies of a complete factorial. Statistically this geometric property implies that all the main-effects and interactions of any t factors are estimable when the other factors are ignored.

Box and Tyssedal (14) showed that a question of importance in factor screening is when a two-level orthogonal design for a multifactor experiment can be projected into lower dimension, typically P=2 or P=3. New results relate to the projectivity P of saturated designs in which n-1 factors are tested in n runs. It is shown that: a design obtained by 'doubling' an $n \times n$ orthogonal array is always of projectivity P=2; a two level cyclic design is either a factorial array, and hence has P=2, or it has P=3; a two level orthogonal design with 4m runs, m odd, has P=3. In particular, these results allow the designs derived by Plackett and Burman (36) to be categorized in terms of these projective properties.

Rosenbaum (44) dispersion experiments examine the effects of control factors on dispersion introduced by error factors. Error factors can be controlled in the laboratory but cannot be controlled when a product is used. For dispersion experiments, Taguchi used direct product designs in which control-factor levels are set by one orthogonal array, error-factor levels are set by another orthogonal array, and, for each combination of control-factor and error-factor levels, an observation is taken. In contrast, compound designs, which include product designs as special cases, share several of the attractive properties of Taguchi's designs, but they are often smaller for given strength, or stronger for a given size, where strength refers to strength of the orthogonal array. Rosenbaum discussed several useful compound designs for dispersion experiments and several general methods for constructing such designs. In particular, compound designs can be built from existing tabulations of confounded factorial designs.

Hedayat and Stufken (26) compound orthogonal arrays have been introduced as an alternative to Taguchi's direct product arrays for studying location and dispersion effect simultaneously. They provided a catalogue of two level compound orthogonal arrays for parameters of most practical interest. The arrays presented possess the maximum possible strength for given numbers of factors and runs. Simple methods to obtain these arrays, using defining contrasts, are also presented.

2. CONCLUSION

Orthogonal arrays are useful in various problems experimental designs. In this study the research and literature review were organized according to the construction and subject.

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